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RESOLVENT NORM DECAY DOES NOT CHARACTERIZE NORM CONTINUITY

BY

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ABSTRACT

We show that there exists a reflexive Banach space $(\mathcal{X}, \|.\|)$ and a strongly continuous semigroup $(\mathcal{T}(t))_{t\geq 0}$ with generator $(\mathcal{A}, D(\mathcal{A}))$ on $(\mathcal{X}, \|.\|)$ such that $\lim_{\mu\in\mathbb{R},\ |\mu|\to\infty} \|R(\mathbf{i}\mu, \mathcal{A})\| = 0$ but $(\mathcal{T}(t))_{t\geq 0}$ is not eventually norm continuous. This answers a question of Amnon Pazy in the negative.

1. Introduction

It is the fundamental principle of semigroup theory that the behavior of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space X and the properties of its generator $(A, D(A))$, or equivalently the properties of the resolvent function $R(\lambda, A) = (\lambda - A)^{-1}$ $(\lambda \in \mathbb{C})$, should closely correlate. Indeed, the Laplace transform carries the regularity properties of the semigroup to the resolvent function while the several inversion and approximation formulas (Trotter-Kato, Post-Widder, etc.) allow the reconstruction of the semigroup from the resolvent (see [7], [17] or [18] for the relevant techniques and results in semigroup theory). And the correlation is indeed perfect if the participants are so: the analytic semigroups, the differentiable or merely eventually differentiable

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semigroups are characterized by the geometry of the spectrum of A and by the growth rate of $R(\lambda, A)$ (see [7, Chapter II.4] or [17, Chapter 2.4 and Chapter 2.5]); and if X happens to be a Hilbert space the characterization of eventual and immediate norm continuity is also known.

THEOREM 1: *Let A be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Hilbert space $(H, \|.\|)$. Let $s(A) = \sup\{\mathrm{Re}\lambda\colon \lambda \in \sigma(A)\}$ denote the spectral bound of A. Then*

1. *$(T(t))_{t\geq 0}$ is immediately norm continuous, i.e., the mapping*

$$T\colon (0, \infty) \to B(H)$$

is continuous, if and only if for some $\mu_0 > s(A)$,

$$\lim_{\mu\in\mathbb{R},\ |\mu|\to\infty} \|R(\mu_0 + \mathbf{i}\mu, A)\| = 0; \tag{1}$$

2. *$(T(t))_{t\geq 0}$ is eventually norm continuous, i.e., there is a $t_0 > 0$ such that the mapping $T\colon (t_0, \infty) \to B(H)$ is continuous, if and only if there is an $n \in \mathbb{N}$ and a $\mu_0 > s(A)$ such that*

$$\lim_{\mu\in\mathbb{R},\ |\mu|\to\infty} \|R(\mu_0 + \mathbf{i}\mu, A)^n T(t_0)\| = 0. \tag{2}$$

Moreover, conditions (1) and (2) are necessary for immediate and eventual norm continuity in arbitrary Banach spaces.

The first part of this result, concerning immediate norm continuity, was first proved by P. You [20]. Later his proof was analyzed, simplified and extended to eventual norm continuity by G.-Q. Xu in [19], by O. El-Mennaoui and K.-J. Engel in [5] and [6], and by O. Blasco, and J. Martínez in [2].

However, even in Hilbert spaces one has to face pathologies. As shown by a counterexample (see e.g. [7, 3.4 Counterexample pp. 273] or [18, Example 1.2.4 pp. 12]), the Spectral Mapping Theorem may fail in Hilbert spaces: if

$$\omega_0 = \inf\{\omega \in \mathbb{R}\colon \exists M_\omega \geq 1\ (\|T(t)\| \leq M_\omega e^{\omega t}\ (t \geq 0))\}$$

denotes the growth bound of $(T(t))_{t\geq 0}$, one can have $s(A) \leq 0 < 1 \leq \omega_0$, in particular $\sigma(T(t)) \setminus \{0\} \neq e^{t\sigma(A)}$. Since the Spectral Mapping Theorem is crucial for the study of the asymptotic behavior of semigroups it is an important revelation that the Spectral Mapping Theorem still holds in arbitrary Banach spaces for **eventually norm continuous** semigroups. Hence it would be of utmost importance to find the characterization of the eventual or at least of the immediate norm continuity of semigroups in Banach spaces.

In the past twenty years a huge amount of research activity has been carried out in order to prove the eventual norm continuity (or at least some weaker norm continuity) of semigroups arising from various PDE's on various Banach spaces (see e.g. [1], [3], [8], [10], [12], [13], [14], [15] and [16] for a representative list of different approaches); a remarkable result of V. Goersmeyer and L. Weis [11] gives that the characterization in Theorem 1.1 holds for positive semigroups in L^p spaces ($1 < p < \infty$). But since neither a theorem analogous to Theorem 1 valid for arbitrary semigroups and Banach spaces was proved nor the optimistic conjecture, often called Amnon Pazy's Question in the literature, that Theorem 1 may hold in every Banach space, was disproved, the results did not give a unified theory of norm continuity and were shadowed by the suspicion of being superfluous.

The purpose of this paper is to show that the problem of characterizing the norm continuity of semigroups in arbitrary Banach spaces indeed necessitates new approaches. We give a negative answer to Pazy's Question, i.e., by constructing a suitable Banach space and a semigroup we show that Theorem 1 does not characterize norm continuity in arbitrary Banach spaces.

THEOREM 2: *There exists a reflexive Banach space* $(\mathcal{X}, \|.\|)$ *and a strongly continuous semigroup* $(\mathcal{T}(t))_{t\geq 0}$ *with generator* $(\mathcal{A}, D(\mathcal{A}))$ *satisfying the following.*

1. $(\mathcal{T}(t))_{t\geq 0}$ *is contractive, i.e.,* $\|\mathcal{T}(t)\| \leq 1$ $(t \geq 0)$*;*
2. $R(\lambda, \mathcal{A})$ *exists for every* $\lambda \in \mathbb{C}$ *with* $\mathrm{Re}\lambda \geq -1$ *and*

$$\lim_{\mu\in\mathbb{R},\ |\mu|\to\infty} \|R(\mathbf{i}\mu, \mathcal{A})\| = 0;$$

3. $(\mathcal{T}(t))_{t\geq 0}$ *is not eventually norm continuous.*

We remark that in our construction the Banach space $(\mathcal{X}, \|.\|)$ is responsible for the pathological behavior while the semigroup $(\mathcal{T}(t))_{t\geq 0}$ is in some sense the simplest possible. After the construction we will discuss the structure of $(\mathcal{X}, \|.\|)$ but we state in advance that $(\mathcal{X}, \|.\|)$ is not a UMD space, in particular it is not L^p for $1 < p < \infty$. Thus a counterexample, if exists, is still lacking in UMD spaces. We will also discuss the failure of the Spectral Mapping Theorem for our semigroup.

Finally we would like to thank Prof. Lutz Weis for pointing out certain particularities of our construction, and in general for his suggestions and helpful remarks.

2. The construction

In this section we construct a family of semigroups and Banach spaces exhibiting more and more pathological behavior. They serve as building blocks for our $(\mathcal{T}(t))_{t\geq 0}$ and $(\mathcal{X}, \|.\|)$. Our reference for the basic notions of semigroup theory is [7]. We recall some definitions related to norm continuity.

Definition 3: A strongly continuous semigroup $(T(t))_{t\geq 0}$ is **eventually norm continuous** if there exists a $t_0 \geq 0$ such that the map $t \mapsto T(t)$ is continuous with respect to the uniform operator topology for $t > t_0$. The semigroup is **immediately norm continuous** if $t_0 = 0$ can be chosen.

The semigroup is **norm continuous at infinity** if

$$\lim_{t\to\infty} \left(\limsup_{h\searrow 0} e^{-\omega_0 t} \|T(t+h) - T(t)\| \right) = 0,$$

where ω_0 is the growth bound of $(T(t))_{t\geq 0}$.

We denote by $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{C}$ the nonnegative integers, the reals, the nonnegative reals and the complex numbers, respectively; log stands for the logarithm to base e.

If X is a product space of n terms or a direct sum over $\mathbb{N}$ we index the coordinates of X starting by 0 and for an $x \in X$ and $j \in \mathbb{N}$, $x(j)$ stands for the j-th coordinate of x. The indexing of matrices also starts by 0, and for an $n \times n$ matrix A, $A(i,j)$ stands for the element in the i-th row and j-th column of A $(0 \leq i, j < n)$.

2.1. The idea. In order to have a Banach space and a semigroup for which the characterization in Theorem 1 fails it is a natural approach to check whether a matrix semigroup $T(t) = e^{At}$ on a finite dimensional space X can fulfill the requirements i) to be contractive (this allows us to glue semigroups together); ii) to have small resolvent along the imaginary axis; iii) to have $\|T(1)x - T(1-\varepsilon)x\| \geq 1/2$ for some $x \in X$ with $\|x\| = 1$ and ε small.

Now it is tempting to take a simple norm on X for which the construction is likely to work according to the current theory of norm continuity, say l^1 norm or l^∞ norm. Unfortunately this approach has been not fruitful yet. Another possibility is to start with the construction of the semigroup. Since one can assume that A has no nontrivial invariant subspaces in X, and we have no candidate for a "nice" norm in X anyway, we can fix a base in X in which A becomes a Jordan block. Then at least we can write up the generator, the semigroup and the resolvents explicitly. It remains to collect (keeping i)-iii) in mind) all the vectors in X that we would like to have norm one; to define the norm on X as the absolute convex hull of these vectors; and to hope that with a bit of luck these vectors are independent enough to have norm one at the same time. Indeed, this is what happens.

2.2. The semigroups. In this section we define semigroups on finite dimensional vector spaces and compute their resolvents. These semigroups will serve as building blocks in our construction.

Let $X_n = \mathbb{C}^{n+1}$. Let $A_n \in \mathbb{C}^{(n+1)\times(n+1)}$ be the Jordan block with eigenvalue $-n$, i.e.,

$$A_n = \begin{pmatrix} -n & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -n & 1 & \dots & 0 & 0 & 0 \\ 0 & 0 & -n & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -n & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & -n & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 & -n \end{pmatrix}.$$

We define $T_n = e^{A_n t}$, so we have (see e.g. [7, Example 2.5. p. 9]

$$T_n(t) = e^{-nt} \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \dots & \frac{t^j}{j!} & \dots & \frac{t^{n-1}}{(n-1)!} & \frac{t^n}{n!} \\ 0 & 1 & t & \dots & \frac{t^{j-1}}{(j-1)!} & \dots & \frac{t^{n-2}}{(n-2)!} & \frac{t^{n-1}}{(n-1)!} \\ 0 & 0 & 1 & \dots & \frac{t^{j-2}}{(j-2)!} & \dots & \frac{t^{n-3}}{(n-3)!} & \frac{t^{n-2}}{(n-2)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & \dots & \frac{t^{n-j-1}}{(n-j-1)!} & \frac{t^{n-j}}{(n-j)!} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & \dots & 1 & t \\ 0 & 0 & 0 & \dots & 0 & \dots & 0 & 1 \end{pmatrix}.$$

The resolvent function of A_n is

$$R(\mathbf{i}\mu, A_n) = (\mathbf{i}\mu - A_n)^{-1} =$$

$$\left(\begin{matrix}
\frac{1}{\mathbf{i}\mu+n} & \frac{1}{(\mathbf{i}\mu+n)^2} & \frac{1}{(\mathbf{i}\mu+n)^3} & \cdots & \frac{1}{(\mathbf{i}\mu+n)^{k+1}} & \cdots \\
0 & \frac{1}{\mathbf{i}\mu+n} & \frac{1}{(\mathbf{i}\mu+n)^2} & \cdots & \frac{1}{(\mathbf{i}\mu+n)^{k}} & \cdots \\
0 & 0 & \frac{1}{\mathbf{i}\mu+n} & \cdots & \frac{1}{(\mathbf{i}\mu+n)^{k-1}} & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\
0 & 0 & 0 & \cdots & \frac{1}{\mathbf{i}\mu+n} & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\
0 & 0 & 0 & \cdots & 0 & \cdots \\
0 & 0 & 0 & \cdots & 0 & \cdots \\
0 & 0 & 0 & \cdots & 0 & \cdots
\end{matrix} \right.$$

$$\left. \begin{matrix}
\cdots & \frac{1}{(\mathbf{i}\mu+n)^{n-1}} & \frac{1}{(\mathbf{i}\mu+n)^{n}} & \frac{1}{(\mathbf{i}\mu+n)^{n+1}} \\
\cdots & \frac{1}{(\mathbf{i}\mu+n)^{n-2}} & \frac{1}{(\mathbf{i}\mu+n)^{n-1}} & \frac{1}{(\mathbf{i}\mu+n)^{n}} \\
\cdots & \frac{1}{(\mathbf{i}\mu+n)^{n-3}} & \frac{1}{(\mathbf{i}\mu+n)^{n-2}} & \frac{1}{(\mathbf{i}\mu+n)^{n-1}} \\
\ddots & \vdots & \vdots & \vdots \\
\cdots & \frac{1}{(\mathbf{i}\mu+n)^{n-k-1}} & \frac{1}{(\mathbf{i}\mu+n)^{n-k}} & \frac{1}{(\mathbf{i}\mu+n)^{n-k+1}} \\
\ddots & \vdots & \vdots & \vdots \\
\cdots & \frac{1}{\mathbf{i}\mu+n} & \frac{1}{(\mathbf{i}\mu+n)^{2}} & \frac{1}{(\mathbf{i}\mu+n)^{3}} \\
\cdots & 0 & \frac{1}{\mathbf{i}\mu+n} & \frac{1}{(\mathbf{i}\mu+n)^{2}} \\
\cdots & 0 & 0 & \frac{1}{\mathbf{i}\mu+n}
\end{matrix} \right).$$

We need to compute the powers of $R(\mathbf{i}\mu, A_n)$. For this, we define the following sequences $(\sigma_k(n))_{n=1}^{\infty}$ $(k \in \mathbb{N})$.

Definition 4: Set $\sigma_0(1) = 1$ and $\sigma_0(n) = 0$ $(1 < n < \infty)$; if $\sigma_{k-1}(n)$ $(1 \leq n < \infty)$ is already defined let

$$\sigma_k(n) = \sum_{j=1}^{n} \sigma_{k-1}(j) \ (1 \leq n < \infty).$$

Lemma 5: *For every $1 \leq m < \infty$ we have*

$$
R(\mathbf{i}\mu, A_n)^m = (\mathbf{i}\mu - A_n)^{-m} =
$$

$$
\begin{pmatrix}
\frac{\sigma_m(1)}{(\mathbf{i}\mu+n)^m} & \frac{\sigma_m(2)}{(\mathbf{i}\mu+n)^{m+1}} & \frac{\sigma_m(3)}{(\mathbf{i}\mu+n)^{m+2}} & \cdots & \frac{\sigma_m(j+1)}{(\mathbf{i}\mu+n)^{m+j}} & \cdots & \cdots & \frac{\sigma_m(n-1)}{(\mathbf{i}\mu+n)^{m+n-2}} & \frac{\sigma_m(n)}{(\mathbf{i}\mu+n)^{m+n-1}} & \frac{\sigma_m(n+1)}{(\mathbf{i}\mu+n)^{m+n}} \\
0 & \frac{\sigma_m(1)}{(\mathbf{i}\mu+n)^m} & \frac{\sigma_m(2)}{(\mathbf{i}\mu+n)^{m+1}} & \cdots & \frac{\sigma_m(j)}{(\mathbf{i}\mu+n)^{m+j-1}} & \cdots & \cdots & \frac{\sigma_m(n-2)}{(\mathbf{i}\mu+n)^{m+n-3}} & \frac{\sigma_m(n-1)}{(\mathbf{i}\mu+n)^{m+n-2}} & \frac{\sigma_m(n)}{(\mathbf{i}\mu+n)^{m+n-1}} \\
0 & 0 & \frac{\sigma_m(1)}{(\mathbf{i}\mu+n)^m} & \cdots & \frac{\sigma_m(j-1)}{(\mathbf{i}\mu+n)^{m+j-2}} & \cdots & \cdots & \frac{\sigma_m(n-3)}{(\mathbf{i}\mu+n)^{m+n-4}} & \frac{\sigma_m(n-2)}{(\mathbf{i}\mu+n)^{m+n-3}} & \frac{\sigma_m(n-1)}{(\mathbf{i}\mu+n)^{m+n-2}} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \frac{\sigma_m(1)}{(\mathbf{i}\mu+n)^m} & \cdots & \cdots & \frac{\sigma_m(n-j-1)}{(\mathbf{i}\mu+n)^{m+n-j-2}} & \frac{\sigma_m(n-j)}{(\mathbf{i}\mu+n)^{m+n-j-1}} & \frac{\sigma_m(n-j+1)}{(\mathbf{i}\mu+n)^{m+n-j}} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & \cdots & \cdots & \frac{\sigma_m(1)}{(\mathbf{i}\mu+n)^m} & \frac{\sigma_m(2)}{(\mathbf{i}\mu+n)^{m+1}} & \frac{\sigma_m(3)}{(\mathbf{i}\mu+n)^{m+2}} \\
0 & 0 & 0 & \cdots & 0 & \cdots & \cdots & 0 & \frac{\sigma_m(1)}{(\mathbf{i}\mu+n)^m} & \frac{\sigma_m(2)}{(\mathbf{i}\mu+n)^{m+1}} \\
0 & 0 & 0 & \cdots & 0 & \cdots & \cdots & 0 & 0 & \frac{\sigma_m(1)}{(\mathbf{i}\mu+n)^m}
\end{pmatrix}.
$$

Proof. We prove the statement by induction on m. For $m = 1$ we have $\sigma_1(n) = 1$ $(1 \leq n < \infty)$ so the statement holds. Suppose that it is true for m. For $0 \leq j \leq k \leq n$ we have

$$
\begin{aligned}
\left[R(\mathbf{i}\mu, A_n)^{m+1}\right](j,k) &= \left[R(\mathbf{i}\mu, A_n)^m R(\mathbf{i}\mu, A_n)\right](j,k) \\
&= \sum_{l=j}^{k} \frac{\sigma_m(l+1-j)}{(\mathbf{i}\mu+n)^{m+l-j}} \frac{1}{(\mathbf{i}\mu+n)^{k+1-l}} \\
&= \frac{\sigma_{m+1}(k+1-j)}{(\mathbf{i}\mu+n)^{m+1+k-j}},
\end{aligned}
$$

while for $0 \leq k < j \leq n$ we have $\left[R(\mathbf{i}\mu, A_n)^{m+1}\right](j,k) = 0$, so the proof is complete. ∎

If $k \in \mathbb{N}$, $\mu \in \mathbb{R}^k$ and $\nu \in \mathbb{N}^k$ are arbitrary, for notational convenience we introduce

$$R\left(\mathbf{i}\mu, A_n\right)^{\nu} = R\left(\mathbf{i}\mu(0), A_n\right)^{\nu(0)} R\left(\mathbf{i}\mu(1), A_n\right)^{\nu(1)}, \ldots, R\left(\mathbf{i}\mu(k-1), A_n\right)^{\nu(k-1)}.$$

Observe that all the $R(\mathbf{i}\mu, A_n)$s commute so the order of our products does not matter. For $\mu \in \mathbb{R}^k$ and $\nu \in \mathbb{N}^k$ we set $|\mu| = |\mu(0)| + \cdots + |\mu(k-1)|$ and $|\nu| = \nu(0) + \cdots + \nu(k-1)$.

Next we show that $R(0, A_n)^{\nu}$ is the biggest of all the $R(\mathbf{i}\mu, A_n)^{\nu}$s.

Lemma 6: *For every $k \in \mathbb{N}$, $\mu \in \mathbb{R}^k$, $\nu \in \mathbb{N}^k$ and $0 \leq j, l \leq n$ we have*

$$\left|\left[R\left(\mathbf{i}\mu, A_n\right)^{\nu}\right](j,l)\right| \leq \left[R(0, A_n)^{|\nu|}\right](j,l).$$

Proof. We prove the statement by induction on $|\nu|$. For $|\nu| = 1$ the statement is obvious. Suppose that it is true for $|\nu| = m$. Take a ν with $|\nu| = m+1$; we can assume that $\nu(0) > 0$. Let ν' be defined by $\nu'(0) = \nu(0) - 1$ and $\nu'(i) = \nu(i)$ for $i \geq 1$. By the induction hypothesis,

$$\begin{aligned}\left|\left[R\left(\mathbf{i}\mu, A_n\right)^{\nu}\right](j,l)\right| &= \left|\left[R(\mathbf{i}\mu(0), A_n) R\left(\mathbf{i}\mu, A_n\right)^{\nu'}\right](j,l)\right| \\ &\leq \left[R(0, A_n) R\left(0, A_n\right)^{m}\right](j,l) = \left[R\left(0, A_n\right)^{m+1}\right](j,l),\end{aligned}$$

as stated. ∎

We close this section with estimates on $\sigma_m(n)$.

Lemma 7: *For $1 \leq m, n < \infty$ we have*

$$\frac{n^{m-1}}{(m-1)!} \leq \sigma_m(n) \leq \frac{(n+m-2)^{m-1}}{(m-1)!}. \tag{3}$$

Proof. We prove the statement by induction on m. Since $\sigma_1(n) = 1$ $(1 \leq n < \infty)$ the statement is true for $m = 1$. Suppose that (3) holds for m. Then by Definition 4,

$$\frac{n^m}{m!} = \int_0^n \frac{x^{m-1}}{(m-1)!}\mathrm{d}x \leq \sum_{j=1}^{n} \frac{j^{m-1}}{(m-1)!} \leq \sum_{j=1}^{n} \sigma_m(j) = \sigma_{m+1}(n) \quad \text{and}$$

$$\begin{aligned}\sigma_{m+1}(n) = \sum_{j=1}^{n} \sigma_m(j) \leq \sum_{j=1}^{n} \frac{(j+m-2)^{m-1}}{(m-1)!} &\leq \int_1^{n+1} \frac{(x+m-2)^{m-1}}{(m-1)!}\mathrm{d}x \\ &\leq \frac{(n+m-1)^m}{m!},\end{aligned}$$

as required. ∎

In the sequel, $\lfloor . \rfloor$ stands for lower integer part.

LEMMA 8: *For* $400 \le n < \infty$,

$$\left(1+\frac{\lfloor\sqrt{n}\rfloor}{n-\lfloor\sqrt{n}\rfloor}\right)^{n-\lfloor\sqrt{n}\rfloor} e^{-\lfloor\sqrt{n}\rfloor} \le \frac{2}{3}$$

and

$$\left(1-\frac{\lfloor\sqrt{n}\rfloor}{n}\right)^{n} e^{\lfloor\sqrt{n}\rfloor} \le \frac{2}{3}.$$

Proof. By taking logarithm, for $400 \le n < \infty$ we get

$$\begin{aligned}
(n-&\lfloor\sqrt{n}\rfloor)\log\left(1+\frac{\lfloor\sqrt{n}\rfloor}{n-\lfloor\sqrt{n}\rfloor}\right)-\lfloor\sqrt{n}\rfloor\\
&\le (n-\lfloor\sqrt{n}\rfloor)\left(\frac{\lfloor\sqrt{n}\rfloor}{n-\lfloor\sqrt{n}\rfloor}-\frac{\lfloor\sqrt{n}\rfloor^2}{2(n-\lfloor\sqrt{n}\rfloor)^2}+\frac{\lfloor\sqrt{n}\rfloor^3}{3(n-\lfloor\sqrt{n}\rfloor)^3}\right)-\lfloor\sqrt{n}\rfloor\\
&=-\frac{\lfloor\sqrt{n}\rfloor^2}{2(n-\lfloor\sqrt{n}\rfloor)}+\frac{\lfloor\sqrt{n}\rfloor^3}{3(n-\lfloor\sqrt{n}\rfloor)^2}\le-\frac{9}{20},
\end{aligned}$$

and

$$\begin{aligned}
n\log\left(1-\frac{\lfloor\sqrt{n}\rfloor}{n}\right)+\lfloor\sqrt{n}\rfloor \le n\left(-\frac{\lfloor\sqrt{n}\rfloor}{n}-\frac{\lfloor\sqrt{n}\rfloor^2}{2n^2}\right)+\lfloor\sqrt{n}\rfloor &= -\frac{\lfloor\sqrt{n}\rfloor^2}{2n}\\
&\le -\frac{9}{20}
\end{aligned}$$

so the statements follows from $e^{-9/20} \le 2/3$. ∎

LEMMA 9: *For every* $0 \le k < \infty$ *and* $\max\{1,k\} \le n < \infty$,

$$e^{n-k}(1-k/n)^{k-n}e^{-nt}t^{n-k} \le \quad 1 \ (t \ge 0). \tag{4}$$

For $t \ge 0$ *the function* $t \mapsto e^{n-k}\left(1-k/n\right)^{k-n}e^{-nt}t^{n-k}$ *attains its maximum 1 for* $t = 1-k/n$*, it is strictly increasing on* $[0, 1-k/n]$ *and strictly decreasing on* $[1-k/n, \infty)$*. Moreover, if* $n \ge e^9$, $0 \le k \le 2n/3$ *and* $|t-(1-k/n)| \ge 2\sqrt{\log(n)/n}$ *then*

$$e^{n-k}(1-k/n)^{k-n}e^{-nt}t^{n-k} \le 1/n. \tag{5}$$

Proof. A simple derivation shows the monotonicity properties and (4) immediately follow. For (5), by monotonicity it is enough to show

$$e^{n-k}\left(1-k/n\right)^{k-n}e^{-nt}t^{n-k} \le 1/n \quad \text{for } |t-(1-k/n)| = 2\sqrt{\log(n)/n}.$$

Let $\tau = \pm 2\sqrt{n\log(n)}$, i.e., $t = 1 - k/n + \tau/n$. Then we have to show that

$$e^{n-k}(1-k/n)^{k-n}e^{-nt}t^{n-k} = e^{-\tau}(1+\tau/(n-k))^{n-k} \leq 1/n.$$

After taking logarithm and using $n/3 \leq n-k \leq n$ we get

$$\begin{aligned} -\tau + (n-k)\log\left(1+\frac{\tau}{n-k}\right) &\leq -\frac{\tau^2}{2(n-k)} + \frac{\tau^3}{3(n-k)^2} \\ &\leq -\frac{4n\log(n)}{2n} + \frac{3^2\cdot 8(n\log(n))^{3/2}}{3n^2} \\ &\leq -2\log(n) + 24\sqrt{\log(n)^3/n} \leq -\log(n) \end{aligned}$$

for $n \geq e^9$. This completes the proof. ∎

The properties in the following definition play a key role in the construction of our Banach spaces.

Definition 10: We say that a sequence $(\gamma_n)_{n\in\mathbb{N}}$ satisfies

(G1) if $\gamma_n \geq 1$ $(n \in \mathbb{N})$;
(G2) if $2(1+2\log(\gamma_{n+d}))e^{1+2\log(\gamma_{n+d})} \leq \log(n)$ $(0 \leq d \leq \sqrt{2n},\ e^6 \leq n < \infty)$;
(G3) if $\gamma_{n+d} \leq \log(n)$ $(0 \leq d \leq \sqrt{2n},\ e^6 \leq n < \infty)$;
(G4) if $\gamma_n \to \infty$ as $n \to \infty$.

The reason for taking $n \geq e^6$ is that even for $\gamma_{n+d} = 1$, (G2) can be satisfied only for $n \gtrsim e^6$. It is easy to see that a sequence satisfying (G1–G4) exists, e.g. $\gamma_n = \max\{1, \log(\log(n))\}$ works, but we do not need an explicit formula for the γ_ns so we omit the elementary computation.

We need two simple observations.

Lemma 11: *For every sequence* $(\gamma_n)_{n\in\mathbb{N}}$,

1. *(G2) implies* $4 + 4\log(\gamma_{n+d}) \leq \log(n)$ $(0 \leq d \leq \sqrt{2n},\ e^6 \leq n < \infty)$*;*
2. *(G3) implies* $3\log(3n)^{1/2}(1+2\log(\gamma_{n+d})) \leq \log(n)$ $(0 \leq d \leq \sqrt{2n},$ $e^{2500} \leq n < \infty)$.

Proof. The statements follow from $4 + 4x \leq 2(1+2x)e^{1+2x}$ $(x \in \mathbb{R}^+)$ and

$$3\log(3n)^{1/2}(1+2\log(\log(n))) \leq \log(n) \quad (e^{2500} \leq n \leq \infty).$$ ∎

Lemma 12: *Let* $e^{60} \leq n < \infty$, $0 \leq d \leq \sqrt{2n}$ *and* $1 \leq m < \infty$. *If* $(\gamma_n)_{n\in\mathbb{N}}$ *satisfies (G1) and (G2), then*

$$\frac{\sigma_m(n+1)}{n^m}\left(\frac{e}{n}\right)^n n!\gamma_{n+d}^m \leq 1 \tag{6}$$

while if $\sqrt{n}+1 \le m < \infty$, *in addition, then*

$$\frac{\sigma_m(n+1)}{n^m}\left(\frac{e}{n}\right)^n n!\gamma_{n+d}^m \le \frac{1}{\sqrt{m-1}}. \tag{7}$$

If $e^{2500} \le n < \infty$ *and* $(\gamma_n)_{n\in\mathbb{N}}$ *satisfies (G1) and (G3), then for* $1 \le m \le \sqrt{\log(3n)}$ *we have*

$$\frac{\sigma_m(n+1)}{n^m}\left(\frac{e}{n}\right)^n n!\gamma_{n+d}^m \le \frac{1}{n^{1/6}}. \tag{8}$$

Proof. By the Stirling formula,

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n+1}} \le n! \le \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n}} \quad (1 \le n < \infty). \tag{9}$$

For $m=1$ and $m=2$ we have $\sigma_1(n)=1$ and $\sigma_2(n)=n$ $(n\in\mathbb{N})$ so (6) and (8) turn to

$$\left(\frac{e}{n}\right)^n n! \le \frac{n}{\gamma_{n+d}},\ \left(\frac{e}{n}\right)^n n! \le \frac{n^2}{(n+1)\gamma_{n+d}^2} \tag{10}$$

and

$$\left(\frac{e}{n}\right)^n n! \le \frac{n^{5/6}}{\gamma_{n+d}},\ \left(\frac{e}{n}\right)^n n! \le \frac{n^{11/6}}{(n+1)\gamma_{n+d}^2}. \tag{11}$$

By (9) both (G2) and (G3) imply

$$\left(\frac{e}{n}\right)^n n! \le \sqrt{2\pi n}e^{1/(12n)} \le \frac{n^{2/3}}{2\log^2(n)} \le \min\left\{\frac{n^{5/6}}{\gamma_{n+d}}, \frac{n^{11/6}}{(n+1)\gamma_{n+d}^2}\right\}$$

for our n and d so (10) and (11) hold.

For $m \ge 3$, by Lemma 7 and (9), the left handside of (6), (7) and (8) can be estimated as

$$\begin{aligned}\frac{\sigma_m(n+1)}{n^m}\left(\frac{e}{n}\right)^n n!\gamma_{n+d}^m &\le \frac{(n+m-1)^{m-1}}{n^m(m-1)!}\left(\frac{e}{n}\right)^n\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n}}\gamma_{n+d}^m \\ &\le \frac{(n+m-1)^{m-1}}{n^m}\frac{1}{\sqrt{2\pi(m-1)}}\left(\frac{e}{m-1}\right)^{m-1} \\ &\quad \times e^{-\frac{1}{12(m-1)+1}}\sqrt{2\pi n}e^{\frac{1}{12n}}\gamma_{n+d}^m \\ &\le \frac{(n+m-1)^{m-1}}{n^{m-1}(m-1)^{m-1}}\frac{1}{\sqrt{n(m-1)}}e^{m-1}\gamma_{n+d}^m e^{\frac{1}{12n}}.\end{aligned}$$

By taking logarithm and replacing $m-1$ with m we have to show that

$$m\log(n+m)+m+(m+1)\log(\gamma_{n+d})+\frac{1}{12n} \tag{12}$$
$$\leq \begin{cases} \left(m+\frac{1}{2}\right)(\log(n)+\log(m)) & \text{for } 2\leq m<\infty; \\ \left(m+\frac{1}{2}\right)(\log(n)+\log(m))-\frac{1}{6}\log(n) & \text{for } 2\leq m\leq\sqrt{\log(3n)}; \\ \left(m+\frac{1}{2}\right)\log(n)+m\log(m) & \text{for } \sqrt{n}\leq m<\infty. \end{cases}$$

Suppose first that $m\leq\sqrt{n}$. Then $m\log(1+m/n)\leq m^2/n\leq 1$ so

$$\begin{aligned} & m\log(n+m)+m+(m+1)\log(\gamma_{n+d})+\frac{1}{12n} \\ & \qquad \leq m\log(n)+m\log\left(1+\frac{m}{n}\right)+m(1+2\log(\gamma_{n+d}))+0.1 \\ & \qquad \leq m\log(n)+m(1+2\log(\gamma_{n+d}))+1.1. \end{aligned} \tag{13}$$

If $1+2\log(\gamma_{n+d})\leq\log(m)$ then the first inequality of (12) holds by $1.1\leq 1/2\log(n)$. While if $\log(2)\leq\log(m)<1+2\log(\gamma_{n+d})$ then by (G2),

$$m(1+2\log(\gamma_{n+d}))\leq e^{1+2\log(\gamma_{n+d})}(1+2\log(\gamma_{n+d}))\leq\frac{1}{2}\log(n)$$

so the first inequality of (12) follows from $1.1\leq 2\log(2)\leq m\log(m)$.

For $2\leq m\leq\log(3n)^{1/2}$, by (G3) and Lemma 11.2 we have

$$m(1+2\log(\gamma_{n+d}))\leq\log(3n)^{1/2}(1+2\log(\gamma_{n+d}))\leq\frac{1}{3}\log(n)$$

so the second inequality of (12) follows again from $1.1\leq 2\log(2)\leq m\log(m)$.

Finally suppose that $\sqrt{n}\leq m$, say $m=n^\alpha$ where $\alpha\geq 1/2$. Then the third inequality of (12) turns to

$$\begin{aligned} m\log(n)+m\log(1+n^{\alpha-1})+m\left(1+\frac{m+1}{m}\log(\gamma_{n+d})\right)+\frac{1}{12n} & \\ \leq m\log(n)+\alpha m\log(n)+\frac{1}{2}\log(n), & \end{aligned} \tag{14}$$

so by $1/(12n)\leq 1\leq 1/2\log(n)$ it is enough to show that

$$m\log(n)+m\log(1+n^{\alpha-1})+m(1+2\log(\gamma_{n+d}))\leq m\log(n)+\alpha m\log(n),$$

i.e., that

$$\log(1+n^{\alpha-1})+1+2\log(\gamma_{n+d})\leq\alpha\log(n).$$

For $1/2\leq\alpha\leq 1$ this follows from (G2) and Lemma 11.1 by

$$\log(1+n^{\alpha-1})+1+2\log(\gamma_{n+d})\leq 2+2\log(\gamma_{n+d})\leq 1/2\log(n).$$

For $\alpha > 1$ we have

$$\begin{aligned}\log(1+n^{\alpha-1}) + 1 + 2\log(\gamma_{n+d}) &\leq \log(n^{\alpha-1}) + 2 + 2\log(\gamma_{n+d}) \\ &= \alpha\log(n) - \log(n) + 2 + 2\log(\gamma_{n+d})\end{aligned}$$

so we are done again by (G2) and Lemma 11.1. This completes the proof. ∎

2.3. THE BANACH SPACES. We turn X_n into a Banach space by defining the unit ball of a norm on it. For this, let $v_n = (0,0,\dots,0,1) \in X_n$ and let $(\gamma_n)_{n\in\mathbb{N}}$ satisfy (G1), (G2) and (G4) of Definition 10. We define the closed unit ball of the norm on X_n by

$$B_n(1) = \text{abs co}\left\{\gamma_n^{|\nu|} R\left(\mathbf{i}\mu, A_n\right)^{\nu} T_n(t)v_n \colon k\in\mathbb{N},\ \mu\in\mathbb{R}^k,\ \nu\in\mathbb{N}^k,\ t\in\mathbb{R}^+\right\},$$

where abs co stands for taking the absolute convex hull. We prove first that $B_n(1)$ is indeed the closed unit ball of a norm on X_n.

LEMMA 13: *Endow X_n with the coordinate supremum norm. Then $B_n(1)$ is a compact convex symmetric set in X_n with nonempty interior; in particular it is the closed unit ball of a norm on X_n. We denote this norm by $\|.\|_n$.*

1. *Every $x \in X_n$ with $\|x\|_n = 1$ can be written as*

$$x = \sum_{j=0}^{l} \alpha_j \gamma_n^{|\nu_j|} R\left(\mathbf{i}\mu_j, A_n\right)^{\nu_j} T_n(t_j)v_n \tag{15}$$

   where $0 \leq l \leq n$, $k_j \in \mathbb{N}$ $(j \leq l)$, $(\nu_j,\mu_j,t_j) \in \mathbb{N}^{k_j}\times\mathbb{R}^{k_j}\times\mathbb{R}^+$ $(j\leq l)$ and $\alpha_j \in \mathbb{C}$ $(j \leq l)$ such that $\sum_{j\leq l}|\alpha_j| = 1$.
2. *Conversely, if an $x \in X_n$ can be written as in (15) where $0 \leq l \leq n$, $k_j \in \mathbb{N}$ $(j \leq l)$, $(\nu_j,\mu_j,t_j) \in \mathbb{N}^{k_j}\times\mathbb{R}^{k_j}\times\mathbb{R}^+$ $(j \leq l)$ and $\alpha_j \in \mathbb{C}$ $(j \leq l)$ such that $\sum_{j\leq l}|\alpha_j| \leq 1$ then $\|x\|_n \leq 1$.*
3. *A functional $\varphi\colon X_n \to \mathbb{C}$ attains its norm on a vector in*

$$\left\{\gamma_n^{|\nu|} R\left(\mathbf{i}\mu, A_n\right)^{\nu} T_n(t)v_n \colon k\in\mathbb{N},\ \mu\in\mathbb{R}^k,\ \nu\in\mathbb{N}^k,\ t\in\mathbb{R}^+\right\}. \tag{16}$$

Proof. First we show that the set of (16) is compact in the coordinate supremum norm. For this it is enough to show that the entries of $\gamma_n^{|\nu|} R\left(\mathbf{i}\mu, A_n\right)^{\nu} T_n(t)$ are bounded and tend to zero as $\max\{|\mu|,|\nu|,t\} \to \infty$. It is clear that the entries of $T(t)$, being of the form $e^{-nt}t^k/k!$ $(0 \leq k \leq n)$, are bounded and tend to zero as $t \to \infty$. Consider now the nonzero entries of $\gamma_n^{|\nu|} R\left(\mathbf{i}\mu, A_n\right)^{\nu}$. With $m = |\nu|$

and $0 \le j \le k \le n$, by Lemma 5 and Lemma 6 we have

$$\begin{aligned}\gamma_n^{|\nu|}\left[R\left(\mathbf{i}\mu, A_n\right)^{\nu}\right](j,k) \le & \gamma_n^m\left[R\left(0, A_n\right)^m\right](j,k) = \gamma_n^m \frac{\sigma_m(k+1-j)}{n^{m+1+k-j}} \\ \le & \gamma_n^m \frac{\sigma_m(n+1)}{n^{m+1}}.\end{aligned}$$

From (6) and (7) of Lemma 12, using $(e/n)^n n! \ge \sqrt{2\pi n}$ from (9), we get

$$\gamma_n^m \frac{\sigma_m(n+1)}{n^{m+1}} \le \frac{1}{n} \quad \text{and} \quad \gamma_n^m \frac{\sigma_m(n+1)}{n^{m+1}} \to 0 \quad \text{as} \quad m \to \infty.$$

So the entries of $\gamma_n^{|\nu|} R\left(\mathbf{i}\mu, A_n\right)^{\nu}$ are bounded uniformly in μ and ν and tend to zero as $|\nu| \to \infty$. Since for $|\nu|$ fixed the entries of $\gamma_n^{|\nu|} R\left(\mathbf{i}\mu, A_n\right)^{\nu}$ tend to zero as $|\mu| \to \infty$, the statement follows. Now $B_n(1)$ is the absolute convex hull of the compact set of (16), so it is compact, convex and symmetric. Its interior is nonempty since $\{R(\mathbf{i}j, A_n)v_n \colon 0 \le j \le n\}$ are already linearly independent (see e.g. [9, Exercise 154, p. 29]).

Next we show 1. The norm one vectors of X_n are on the boundary of $B_n(1)$. Since $B_n(1)$ is compact and convex, its boundary points can be obtained as the sum one linear combinations of the extreme points of $B_n(1)$. Now $B_n(1)$ is the absolute convex hull of the compact set of (16) so the extreme points of $B_n(1)$ are in the set of (16). This proves 1.

Statement 2 follows directly from the definition of $B_n(1)$. Since a functional attains its norm on an extreme point of the unit ball we have statement 3, which completes the proof. ∎

We denote by $\|.\|_n$ the operator norm on $B(X_n)$ and also the norm of functionals on X_n.

Lemma 14: *The semigroup $(T_n(t))_{t \ge 0}$ satisfies $\|T_n(t)\|_n \le 1$ $(t \ge 0)$.*

Proof. We show that $\|T_n(t)x\|_n \le 1$ for every $x \in X_n$ with $\|x\|_n = 1$ and $t \ge 0$. Take $x \in X_n$ with $\|x\|_n = 1$. By Lemma 13.1 there are $0 \le l \le n$, $k_j \in \mathbb{N}$ $(j \le l)$, $(\nu_j, \mu_j, t_j) \in \mathbb{N}^{k_j} \times \mathbb{R}^{k_j} \times \mathbb{R}^+$ $(j \le l)$ and $\alpha_j \in \mathbb{C}$ $(j \le l)$ such that $\sum_{j \le l} |\alpha_j| = 1$ and

$$x = \sum_{j=0}^{l} \alpha_j \gamma_n^{|\nu_j|} R\left(\mathbf{i}\mu_j, A_n\right)^{\nu_j} T_n(t_j) v_n.$$

Let $t \geq 0$ be arbitrary. Since the operators in the semigroup and the resolvents commute,

$$
\begin{aligned}
T_n(t)x =& T_n(t)\left(\sum_{j=0}^{l} \alpha_j \gamma_n^{|\nu_j|} R\left(\mathbf{i}\mu_j, A_n\right)^{\nu_j} T_n(t_j)v_n\right) \\
=& \sum_{j=0}^{l} \alpha_j \gamma_n^{|\nu_j|} R\left(\mathbf{i}\mu_j, A_n\right)^{\nu_j} T_n(t+t_j)v_n.
\end{aligned} \tag{17}
$$

Thus by Lemma 13.2, $\|T_n(t)x\|_n \leq 1$. ∎

Lemma 15: *The resolvent $R(\lambda, A_n)$ exists for every $\lambda \in \mathbb{C} \setminus \{-n\}$ and there is a constant $C_n > 0$ such that*

$$\|R(\mathbf{i}\mu, A_n)\|_n \leq \min\{1/\gamma_n, C_n/|\mu|\} \quad (\mu \in \mathbb{R}).$$

Proof. Since A_n is a Jordan block of eigenvalue $-n$, $R(\lambda, A_n)$ exists for every $\lambda \in \mathbb{C} \setminus \{-n\}$. Let $x \in X_n$ satisfy $\|x\|_n = 1$. Then by Lemma 13.1, for some $0 \leq l \leq n$, $k_j \in \mathbb{N}$ $(j \leq l)$, $(\nu_j, \mu_j, t_j) \in \mathbb{N}^{k_j} \times \mathbb{R}^{k_j} \times \mathbb{R}^+$ $(j \leq l)$ and $\alpha_j \in \mathbb{C}$ $(j \leq l)$ we have $\sum_{j \leq l} |\alpha_j| = 1$ and

$$x = \sum_{j=0}^{l} \alpha_j \gamma_n^{|\nu_j|} R\left(\mathbf{i}\mu_j, A_n\right)^{\nu_j} T_n(t_j)v_n.$$

By Lemma 13.2,

$$
\begin{aligned}
R(\mathbf{i}\mu, A_n)x =& \sum_{j=0}^{l} \alpha_j \gamma_n^{|\nu_j|} R\left(\mathbf{i}(\mu_j, \mu), A_n\right)^{(\nu_j, 1)} T_n(t_j)v_n \\
=& \sum_{j=0}^{l} \frac{\alpha_j}{\gamma_n} \gamma_n^{|(\nu_j, 1)|} R\left(\mathbf{i}(\mu_j, \mu), A_n\right)^{(\nu_j, 1)} T_n(t_j)v_n
\end{aligned}
$$

is of norm at most $1/\gamma_n$, which proves the first part of the statement. The existence of C_n follows from the definition $R(\mathbf{i}\mu, A_n) = (\mathbf{i}\mu - A_n)^{-1}$. ∎

Proposition 16: *For $e^{61} \leq n < \infty$, $\|T_n(1)v_n - T_n(1 - \lfloor\sqrt{n}\rfloor/n)v_n\|_n \geq 1/3$.*

Proof. Consider the functional $\varphi\colon X_n \to \mathbb{C}$ defined by

$$\varphi(x) = n! e^n x(0) - (n - \lfloor\sqrt{n}\rfloor)! e^{n - \lfloor\sqrt{n}\rfloor} \left(1 - \lfloor\sqrt{n}\rfloor/n\right)^{-n + \lfloor\sqrt{n}\rfloor} x(\lfloor\sqrt{n}\rfloor).$$

It suffices to show that $\|\varphi\|_n \leq 2$ and that

$$\varphi\left(T_n(1)v_n - T_n(1 - \lfloor\sqrt{n}\rfloor/n)v_n\right) \geq 2/3. \tag{18}$$

To have $\|\varphi\|_n \leq 2$, by Lemma 13.3 it is enough to check $|\varphi(x)| \leq 2$ for every $x = \gamma_n^{|\nu|} R(\mathbf{i}\mu, A_n)^\nu T_n(t)v_n$ where $k \in \mathbb{N}$, $\mu \in \mathbb{R}^k$, $\nu \in \mathbb{N}^k$ and $t \in \mathbb{R}^+$. Let $m = |\nu|$; since $T_n(t)$ has nonnegative entries, by Lemma 6 we have

$$\begin{aligned}|\varphi(x)| \leq & n!e^n \gamma_n^m \left[R(0, A_n)^m T_n(t)v_n\right](0) \\ & + (n - \lfloor\sqrt{n}\rfloor)!e^{n-\lfloor\sqrt{n}\rfloor} \left(1 - \lfloor\sqrt{n}\rfloor/n\right)^{-n+\lfloor\sqrt{n}\rfloor} \\ & \times \gamma_n^m \left[R(0, A_n)^m T_n(t)v_n\right](\lfloor\sqrt{n}\rfloor).\end{aligned}$$

Now if $m = 0$,

$$\left[T_n(t)v_n\right](0) = e^{-nt}\frac{t^n}{n!} \text{ and } \left[T_n(t)v_n\right](\lfloor\sqrt{n}\rfloor) = e^{-nt}\frac{t^{n-\lfloor\sqrt{n}\rfloor}}{(n-\lfloor\sqrt{n}\rfloor)!},$$

while if $m \geq 1$,

$$\begin{aligned}\left[R(0, A_n)^m T_n(t)v_n\right](0) = & \sum_{j=0}^{n} \frac{\sigma_m(n+1-j)}{n^{m+n-j}}\frac{t^j}{j!}e^{-nt} \leq \frac{\sigma_m(n+1)}{n^{m+n}}\sum_{j=0}^{n}\frac{(nt)^j}{j!}e^{-nt} \\ \leq & \frac{\sigma_m(n+1)}{n^{m+n}}\end{aligned}$$

and

$$\begin{aligned}\left[R(0, A_n)^m T_n(t)v_n\right](\lfloor\sqrt{n}\rfloor) = & \sum_{j=0}^{n-\lfloor\sqrt{n}\rfloor} \frac{\sigma_m(n-\lfloor\sqrt{n}\rfloor+1-j)}{n^{m+n-\lfloor\sqrt{n}\rfloor-j}}\frac{t^j}{j!}e^{-nt} \\ \leq & \frac{\sigma_m(n-\lfloor\sqrt{n}\rfloor+1)}{n^{m+n-\lfloor\sqrt{n}\rfloor}}\sum_{j=0}^{n-\lfloor\sqrt{n}\rfloor}\frac{(nt)^j}{j!}e^{-nt} \\ \leq & \frac{\sigma_m(n-\lfloor\sqrt{n}\rfloor+1)}{n^{m+n-\lfloor\sqrt{n}\rfloor}}.\end{aligned} \tag{19}$$

Thus if $m = 0$,

$$\begin{aligned}|\varphi(x)| \leq & n!e^n e^{-nt}\frac{t^n}{n!} + (n-\lfloor\sqrt{n}\rfloor)!e^{n-\lfloor\sqrt{n}\rfloor}\left(1 - \frac{\lfloor\sqrt{n}\rfloor}{n}\right)^{-n+\lfloor\sqrt{n}\rfloor}e^{-nt}\frac{t^{n-\lfloor\sqrt{n}\rfloor}}{(n-\lfloor\sqrt{n}\rfloor)!} \\ = & e^n e^{-nt}t^n + e^{n-\lfloor\sqrt{n}\rfloor}\left(1 - \frac{\lfloor\sqrt{n}\rfloor}{n}\right)^{\lfloor\sqrt{n}\rfloor-n}e^{-nt}t^{n-\lfloor\sqrt{n}\rfloor} \leq 2\end{aligned}$$

by Lemma 9 for $k = 0$ and $k = \lfloor\sqrt{n}\rfloor$. While if $m \geq 1$,

$$
\begin{aligned}
|\varphi(x)| \leq & n! e^n \gamma_n^m \frac{\sigma_m(n+1)}{n^{m+n}} \\
& + (n - \lfloor\sqrt{n}\rfloor)! e^{n-\lfloor\sqrt{n}\rfloor} \left(1 - \lfloor\sqrt{n}\rfloor/n\right)^{-n+\lfloor\sqrt{n}\rfloor} \gamma_n^m \frac{\sigma_m(n - \lfloor\sqrt{n}\rfloor + 1)}{n^{m+n-\lfloor\sqrt{n}\rfloor}} \\
= & n! \left(\frac{e}{n}\right)^n \gamma_n^m \frac{\sigma_m(n+1)}{n^m} \\
& + (n - \lfloor\sqrt{n}\rfloor)! \left(\frac{e}{n - \lfloor\sqrt{n}\rfloor}\right)^{n-\lfloor\sqrt{n}\rfloor} \gamma_{n-\lfloor\sqrt{n}\rfloor+\lfloor\sqrt{n}\rfloor}^m \frac{\sigma_m(n - \lfloor\sqrt{n}\rfloor + 1)}{(n - \lfloor\sqrt{n}\rfloor)^m} \\
& \times \left(1 - \frac{\lfloor\sqrt{n}\rfloor}{n}\right)^m .
\end{aligned}
$$

Since $e^{61} \leq n$ implies $e^{60} \leq n - \lfloor\sqrt{n}\rfloor$ and $\lfloor\sqrt{n}\rfloor \leq \sqrt{2(n - \lfloor\sqrt{n}\rfloor)}$, by (6) in Lemma 12 for $d = 0$ and $d = \lfloor\sqrt{n}\rfloor$ we get

$$
n! \left(\frac{e}{n}\right)^n \gamma_n^m \frac{\sigma_m(n+1)}{n^m} \leq 1
$$

and

$$
(n - \lfloor\sqrt{n}\rfloor)! \left(\frac{e}{n - \lfloor\sqrt{n}\rfloor}\right)^{n-\lfloor\sqrt{n}\rfloor} \gamma_{n-\lfloor\sqrt{n}\rfloor+\lfloor\sqrt{n}\rfloor}^m \frac{\sigma_m(n - \lfloor\sqrt{n}\rfloor + 1)}{(n - \lfloor\sqrt{n}\rfloor)^m} \leq 1;
$$

thus we concluded $|\varphi(x)| \leq 2$.

To show (18), by Lemma 8 we have

$$
\begin{aligned}
& \varphi(T_n(1)v_n) \\
& \quad = n! e^n \frac{e^{-n}}{n!} - (n - \lfloor\sqrt{n}\rfloor)! e^{n-\lfloor\sqrt{n}\rfloor} \left(1 - \lfloor\sqrt{n}\rfloor/n\right)^{-n+\lfloor\sqrt{n}\rfloor} \frac{e^{-n}}{(n - \lfloor\sqrt{n}\rfloor)!} \\
& \quad = 1 - \left(\frac{n}{n - \lfloor\sqrt{n}\rfloor}\right)^{n-\lfloor\sqrt{n}\rfloor} e^{-\lfloor\sqrt{n}\rfloor} \\
& \quad = 1 - \left(1 + \frac{\lfloor\sqrt{n}\rfloor}{n - \lfloor\sqrt{n}\rfloor}\right)^{n-\lfloor\sqrt{n}\rfloor} e^{-\lfloor\sqrt{n}\rfloor} \geq 1/3
\end{aligned}
$$

and

$$\begin{aligned}&\varphi(T_n(1-\lfloor\sqrt{n}\rfloor/n)v_n)\\ =&n!e^n e^{-n+\lfloor\sqrt{n}\rfloor}\frac{(1-\lfloor\sqrt{n}\rfloor/n)^n}{n!}\\ &-(n-\lfloor\sqrt{n}\rfloor)!e^{n-\lfloor\sqrt{n}\rfloor}\left(1-\lfloor\sqrt{n}\rfloor/n\right)^{-n+\lfloor\sqrt{n}\rfloor}e^{-n+\lfloor\sqrt{n}\rfloor}\frac{(1-\lfloor\sqrt{n}\rfloor/n)^{n-\lfloor\sqrt{n}\rfloor}}{(n-\lfloor\sqrt{n}\rfloor)!}\\ =&\left(1-\frac{\lfloor\sqrt{n}\rfloor}{n}\right)^n e^{\lfloor\sqrt{n}\rfloor}-1\leq-\frac{1}{3}\end{aligned}$$

thus

$$\varphi\left(T_n(1)v_n-T_n(1-\lfloor\sqrt{n}\rfloor/n)v_n\right)\geq 2/3,$$

as required. ∎

2.4. The counterexample. Let $1\leq p<\infty$ be arbitrary fixed and set

$$(\mathcal{X},\|.\|)=\bigoplus_{n=e^{61}}^{\infty}(X_n,\|.\|_n)$$

as an l^p-sum of Banach spaces, i.e., for $x\in\mathcal{X}$,

$$\|x\|=\left(\sum_{n=e^{61}}^{\infty}\|x(n)\|_n^p\right)^{1/p}.$$

As usual, $\|.\|$ stands for the norm of operators and functionals on $\mathcal{X}$, as well.

Consider the operators

$$\mathcal{T}(t)=\bigoplus_{n=e^{61}}^{\infty}T_n\left(t/\sqrt{\gamma_n}\right)\quad(t\geq 0),$$

and let the unbounded operator $\mathcal{A}\colon\mathcal{X}\to\mathcal{X}$ be $\mathcal{A}=\bigoplus_{n=e^{61}}^{\infty}\frac{1}{\sqrt{\gamma_n}}A_n$ with natural domain $D(\mathcal{A})=\{x\in\mathcal{X}\colon\mathcal{A}x\in\mathcal{X}\}$. We show that $(\mathcal{X},\|.\|)$, $(\mathcal{A},D(\mathcal{A}))$ and $(\mathcal{T}(t))_{t\geq 0}$ fulfill the requirements of Theorem 2.

Proposition 17: *With the notation introduced above we have the following.*

1. $(\mathcal{X},\|.\|)$ *is a Banach space which is reflexive for* $1<p<\infty$*;*
2. $(\mathcal{T}(t))_{t\geq 0}$ *is a strongly continuous semigroup of bounded operators satisfying* $\|\mathcal{T}(t)\|\leq 1$ $(t\geq 0)$*;*
3. $(\mathcal{A},D(\mathcal{A}))$ *is the generator of* $(\mathcal{T}(t))_{t\geq 0}$*;*
4. $R(\lambda,\mathcal{A})$ *exists for every* $\lambda\in\mathbb{C}\setminus\left\{-k/\sqrt{\gamma_k}\colon e^{61}\leq k<\infty\right\}$ *and*

$$\lim_{\mu\in\mathbb{R},\ |\mu|\to\infty}\|R(\mathbf{i}\mu,\mathcal{A})\|=0;$$

5. $(\mathcal{T}(t))_{t\geq 0}$ *is not eventually norm continuous; moreover* $(\mathcal{T}(t))_{t\geq 0}$ *is not norm continuous at infinity.*

Proof. The first statement is obvious. For the second, by Lemma 14 we have $\|\mathcal{T}(t)|_{X_n}\| \leq 1$ $(t \geq 0, e^{61} \leq n < \infty)$ which implies $\|\mathcal{T}(t)\| \leq 1$ $(t \geq 0)$. It is clear that $(\mathcal{T}(t))_{t\geq 0}$ is a semigroup so it remains to show that it is strongly continuous; by [7, 5.3 Proposition p. 38] it is enough to show $\lim_{t\searrow 0} \mathcal{T}(t)x = x$ for every $x \in \mathcal{X}$. Fix $\varepsilon > 0$ and take an $x \in \mathcal{X}$. There is an $N > e^{61}$ such that $\sum_{n=N+1}^{\infty} \|x(j)\|_j^p \leq (\varepsilon/4)^p$. Since the restriction of $(\mathcal{T}(t))_{t\geq 0}$ to the finite dimensional space $\bigoplus_{n=e^{61}}^{N} X_n$ is norm continuous there is a $\tau > 0$ such that for every $0 \leq t < \tau$ and $x' \in \bigoplus_{n=e^{61}}^{N} X_n$ we have $\|\mathcal{T}(t)x' - x'\| \leq \varepsilon/2$. Let $x = x' + x''$ with $x' \in \bigoplus_{n=e^{61}}^{N} X_n$ and $x'' \in \bigoplus_{n=N+1}^{\infty} X_n$; then $\|x''\| \leq \varepsilon/4$. Since $(\mathcal{T}(t))_{t\geq 0}$ is contractive we have

$$\|\mathcal{T}(t)x - x\| \leq \|\mathcal{T}(t)x' - x'\| + \|\mathcal{T}(t)x'' - x''\| \leq \frac{\varepsilon}{2} + \|\mathcal{T}(t)x''\| + \|x''\| \leq \varepsilon$$

$(0 \leq t < \tau)$, as required.

For 3, observe that $(\mathcal{A}, D(\mathcal{A}))$ is a closed densely defined operator. Since the generator of $\mathcal{T}(t)|_{X_n}$ is $A_n/\sqrt{\gamma_n}$, $\mathcal{A}$ is the generator of $(\mathcal{T}(t))_{t\geq 0}$.

We turn to 4. By Lemma 15, $R(\lambda, A_n/\sqrt{\gamma_n}) = \sqrt{\gamma_n}R(\sqrt{\gamma_n}\lambda, A_n)$ exists for every $\lambda \in \mathbb{C} \setminus \{-n/\sqrt{\gamma_n}\}$ $(1 \leq n < \infty)$. Thus for $\lambda \in \mathbb{C}$ fixed, $R(\lambda, A_n)$ exists for sufficiently large n and by Lemma 15 and (G4) of Definition 10,

$$|\lambda|\|R(0, A_n/\sqrt{\gamma_n})\|_n \leq |\lambda|\sqrt{\gamma_n} \to 0 \quad \text{as } n \to \infty;$$

in particular $\left[I + \lambda R(0, A_n/\sqrt{\gamma_n})\right]^{-1}$ exists and is bounded in norm uniformly in n. So if $\lambda \in \mathbb{C} \setminus \{-k/\sqrt{\gamma_k} \colon e^{61} \leq k < \infty\}$ is fixed, $R(\lambda, A_n/\sqrt{\gamma_n}) = R(0, A_n/\sqrt{\gamma_n})\left[I + \lambda R(0, A_n/\sqrt{\gamma_n})\right]^{-1}$ exists and it is bounded in norm uniformly in n. So we have $R(\lambda, \mathcal{A}) = \bigoplus_{n=e^{61}}^{\infty} R(\lambda, A_n/\sqrt{\gamma_n})$, a bounded operator on $\mathcal{X}$, as required. Moreover, by Lemma 15 we have

$$\|R(\mathbf{i}\mu, \mathcal{A})\| = \sup_{e^{61} \leq n < \infty} \|R(\mathbf{i}\mu, A_n/\sqrt{\gamma_n})\| \leq \sup_{e^{61} \leq n < \infty} \min\left\{\frac{1}{\sqrt{\gamma_n}}, \frac{C_n}{|\mu|}\right\}, \tag{20}$$

and by (G4) of Definition 10,

$$\sup_{e^{61} \leq n < \infty} \min\{1/\sqrt{\gamma_n}, C_n/|\mu|\} \to 0 \quad \text{as } |\mu| \to \infty,$$

which proves 4.

To have 5, we show first that $(\mathcal{T}(t))_{t\geq 0}$ is not norm continuous in any $t_0 \geq 0$. Fix an arbitrary $t_0 \geq 0$. Let $n \geq e^{61}$ such that $t_0 \leq \sqrt{\gamma_n}/2$. Let $x_n \in \mathcal{X}$ be the vector which is nonzero only on X_n and $x_n|_{X_n} = v_n$. Let $\tau_n > 0$ be defined by $t_0 + \tau_n = \sqrt{\gamma_n}(1 - \lfloor\sqrt{n}\rfloor/n)$ and set $y_n = \mathcal{T}(\tau_n)x_n$. Then $\|x_n\| \leq 1$ and $\|\mathcal{T}(\tau_n)\| \leq 1$ imply $\|y_n\| \leq 1$ while by Proposition 16 we have

$$\begin{aligned}\left\|\mathcal{T}\left(t_0 + \frac{\sqrt{\gamma_n}\lfloor\sqrt{n}\rfloor}{n}\right)y_n - \mathcal{T}(t_0)y_n\right\| \\ &= \left\|T_n\left(\frac{t_0+\tau_n}{\sqrt{\gamma_n}} + \frac{\lfloor\sqrt{n}\rfloor}{n}\right)v_n - T_n\left(\frac{t_0+\tau_n}{\sqrt{\gamma_n}}\right)v_n\right\|_n \\ &= \left\|T_n(1)v_n - T_n\left(1 - \frac{\lfloor\sqrt{n}\rfloor}{n}\right)v_n\right\|_n \geq \frac{1}{3}.\end{aligned}$$

By (G2) of Definition 10, $\sqrt{\gamma_n}\lfloor\sqrt{n}\rfloor/n \to 0$ as $n \to \infty$ thus $(\mathcal{T}(t))_{t\geq 0}$ is not norm continuous in t_0, indeed. Moreover, for the growth bound of $(\mathcal{T}(t))_{t\geq 0}$ we have $\omega_0 = 0$ and

$$\limsup_{t\to\infty}\left(\limsup_{h\searrow 0}\|\mathcal{T}(t+h) - \mathcal{T}(t)\|\right) \geq \frac{1}{3},$$

thus $(\mathcal{T}(t))_{t\geq 0}$ is not norm continuous at infinity. This completes the proof. ∎

3. Analysis

To close this paper we would like to point out certain particularities of the construction above. First, we show that, for n sufficiently large, $(X_n, \|.\|_n)$ contains a subspace of dimension $\log(n)$ where the norm is approximately the l^1 norm (see (21)). This shows, in particular, that the resulting Banach space $(\mathcal{X}, \|.\|)$ is not a UMD space (see e.g. [4]). Second, we describe the spectrum of $\mathcal{T}(t)$ $(t \geq 0)$ and we obtain a drastic failure of any spectral mapping theorem. Finally we revisit the decay of $\|R(\mathbf{i}\mu, \mathcal{A})\|$ for $|\mu| \to \infty$.

3.1. The structure of $(\mathcal{X}, \|.\|)$. Our construction has one free parameter: the sequence $(\gamma_n)_{n\in\mathbb{N}}$. It has to be chosen in such a way that (6) in Lemma 12 holds. By Lemma 7 and (9),

$$\begin{aligned}1 \geq& \gamma_n^m \frac{\sigma_m(n+1)}{n^m}\left(\frac{e}{n}\right)^n n! \geq \gamma_n^m \frac{n^{m-1}}{(m-1)!n^m}\left(\frac{e}{n}\right)^n\left(\frac{n}{e}\right)^n\sqrt{2\pi n} \\ \geq& \left(\frac{e\gamma_n}{m-1}\right)^{m-1}\frac{e^{-1/(12(m-1))}}{\sqrt{n(m-1)}}.\end{aligned}$$

In particular, for $m-1=\log(n)$ we get $(\gamma_n/\log(n))^{\log(n)} \le \sqrt{\log(n)/n}$. This shows $\gamma_n \le \log(n)$ $(3 \le n < \infty)$. We will use this observation in two ways: by assuming $\gamma_{n+d} \le \log(n)$ $(3 \le n < \infty\ , 0 \le d < \sqrt{2n})$ and by assuming $\gamma_n/\gamma_{n-k} \le 3/2$ $(n \in \mathbb{N},\ n/2 \le k \le n)$. Observe that (G3) of Definition 10 is then satisfied.

Let $d_n \in \mathbb{N}$ satisfy $4\sqrt{n\log(n)}+1 < d_n < n/(6\log(n))$ and set

$$K_n = \{\lfloor (n+1)/2 \rfloor + jd_n \colon 0 \le j < \log(n)\}.$$

We show that for n sufficiently large, on $Y_n = \operatorname{span}\langle T_n(1-k/n)v_n \colon k \in K_n\rangle \le X_n$ the $\|.\|_n$ norm and the l^1 norm approximately coincide, i.e., for every $y \in Y_n$, say $y = \sum_{k\in K_n} \alpha_k T_n(1-k/n)v_n$ where $\alpha_k \in \mathbb{C}$ $(k \in K_n)$, we have

$$\frac{n-\log(n)}{n+\log(n)} \sum_{k\in K_n} |\alpha_k| \le \|y\|_n \le \sum_{k\in K_n} |\alpha_k|. \tag{21}$$

The inequality $\|y\|_n \le \sum_{k\in K_n} |\alpha_k|$ holds by Lemma 13.2. For the other inequality, consider the functional $\varphi\colon X_n \to \mathbb{C}$,

$$\varphi(x) = \sum_{k\in K_n} (n-k)! e^{n-k} \left(1-k/n\right)^{k-n} \frac{|\alpha_k|}{\alpha_k} x(k).$$

Just as in the proof of Lemma 16, it is enough to show that $\|\varphi\|_n \le 1+\log(n)/n$ and that $\varphi(y) \ge (1-\log(n)/n)\sum_{k\in K_n}|\alpha_k|$.

To have $\|\varphi\|_n \le 1+\log(n)/n$, by Lemma 13.3 it is enough to check $|\varphi(x)| \le 1+\log(n)/n$ for every $x = \gamma_n^{|\nu|} R\left(\mathbf{i}\mu, A_n\right)^{\nu} T_n(t)v_n$ where $k \in \mathbb{N}$, $\mu \in \mathbb{R}^k$, $\nu \in \mathbb{N}^k$ and $t \in \mathbb{R}^+$. Let $m = |\nu|$; since $T_n(t)$ has nonnegative entries, by Lemma 6 we have

$$|\varphi(x)| \le \sum_{k\in K_n} (n-k)! e^{n-k} \left(1-k/n\right)^{k-n} \gamma_n^m \left[R(0,A_n)^m T_n(t)v_n\right](k).$$

If $m=0$ we have $\left[T_n(t)v_n\right](k) = e^{-nt}\frac{t^{n-k}}{(n-k)!}$ while if $m \ge 1$, as we have seen in (19) of the proof of Lemma 16,

$$\left[R(0,A_n)^m T_n(t)v_n\right](k) \le \frac{\sigma_m(n-k+1)}{n^{m+n-k}}.$$

Thus if $m = 0$,

$$|\varphi(x)| \le \sum_{k\in K_n} (n-k)! e^{n-k} \left(1 - \frac{k}{n}\right)^{k-n} e^{-nt} \frac{t^{n-k}}{(n-k)!} \tag{22}$$
$$= \sum_{k\in K_n} e^{n-k} \left(1 - \frac{k}{n}\right)^{k-n} e^{-nt} t^{n-k}.$$

By Lemma 9,

$$e^{n-k} (1 - k/n)^{k-n} e^{-nt} t^{n-k} \le 1 \quad (k \in K_n).$$

Moreover, since $d_n/n \ge 4\sqrt{\log(n)/n}$ and $0 \le k \le 2n/3$ $(k \in K_n)$, by Lemma 9

$$e^{n-k} \left(1 - \frac{k}{n}\right)^{k-n} e^{-nt} t^{n-k} \le \frac{1}{n} \quad (k \in K_n,\ |t - (1 - k/n)| \ge d_n/(2n)),$$

that is for every $k \in K_n$ with at most one exception. Thus for $m = 0$, by (22) we get $|\varphi(x)| \le 1 + \log(n)/n$.

Consider now $1 \le m < \infty$. We have

$$|\varphi(x)| \le \sum_{k\in K_n} (n-k)! e^{n-k} (1 - k/n)^{k-n} \gamma_n^m \frac{\sigma_m(n-k+1)}{n^{m+n-k}} \tag{23}$$
$$= \sum_{k\in K_n} (n-k)! \left(\frac{e}{n-k}\right)^{n-k} \gamma_{n-k}^m \frac{\sigma_m(n-k+1)}{(n-k)^m} \left(\frac{\gamma_n}{\gamma_{n-k}}\right)^m \left(1 - \frac{k}{n}\right)^m.$$

First, Let $1 \le m < \sqrt{\log(n)}$. We have (G3) for our $(\gamma_n)_{n\in\mathbb{N}}$ thus (8) of Lemma 12 can be applied and by $n - k \ge n/3$ $(k \in K_n)$ we get

$$(n-k)!\left(\frac{e}{n-k}\right)^{n-k} \gamma_{n-k}^m \frac{\sigma_m(n-k+1)}{(n-k)^m} \le \frac{1}{(n-k)^{1/6}} \le \frac{2}{n^{1/6}} \quad (k \in K_n).$$

So by (23) and $\gamma_n/\gamma_{n-k} \le 3/2$ $(k \in K_n)$,

$$|\varphi(x)| \le \frac{2}{n^{1/6}} \sum_{k\in K_n} \left(\frac{\gamma_n}{\gamma_{n-k}}\right)^m \left(1 - \frac{k}{n}\right)^m \le \frac{2\log(n)(3/2)^{\sqrt{\log(n)}}}{n^{1/6}} \le 1$$

for n sufficiently large.

For $\sqrt{\log(n)} \le m < \infty$, by (6) in Lemma 12, (23) can be continued as

$$|\varphi(x)| \le \sum_{k\in K_n} (n-k)!\Big(\frac{e}{n-k}\Big)^{n-k} \gamma_{n-k}^m \frac{\sigma_m(n-k+1)}{(n-k)^m}\Big(\frac{\gamma_n}{\gamma_{n-k}}\Big)^m \Big(1-\frac{k}{n}\Big)^m$$
$$\le \sum_{k\in K_n} \Big(\frac{\gamma_n}{\gamma_{n-k}}\Big)^m \Big(1-\frac{k}{n}\Big)^m.$$

We have $\gamma_n/\gamma_{n-k} \le 3/2$ and $1-k/n \le 1/2$ $(k \in K_n)$ so

$$\sum_{k\in K_n} \Big(\frac{\gamma_n}{\gamma_{n-k}}\Big)^m \Big(1-\frac{k}{n}\Big)^m \le \log(n)(3/2)^m(1/2)^m \le \log(n)(3/4)^{\log(n)^{1/2}} \le 1$$

for n sufficiently large. Thus we concluded $|\varphi(x)| \le 1+\log(n)/n$ for n sufficiently large.

Finally to have $\varphi(y) \ge (1-\log(n)/n)\sum_{k\in K_n} |\alpha_k|$,

$$\varphi(y) = \sum_{k\in K_n} (n-k)!e^{n-k}\Big(1-\frac{k}{n}\Big)^{k-n}\frac{|\alpha_k|}{\alpha_k}\Big[\sum_{l\in K_n} \alpha_l T_n(1-l/n)v_n\Big](k)$$
$$\ge \sum_{k\in K_n} |\alpha_k|(n-k)!e^{n-k}\Big(1-\frac{k}{n}\Big)^{k-n}\left[T_n(1-k/n)v_n\right](k)$$
$$- \sum_{k\in K_n} (n-k)!e^{n-k}\Big(1-\frac{k}{n}\Big)^{k-n}\Big[\sum_{l\in K_n,\ l\ne k} |\alpha_l| T_n(1-l/n)v_n\Big](k).$$

As we have seen for the estimate of (22),

$$\sum_{k\in K_n} |\alpha_k|(n-k)!e^{n-k}\Big(1-\frac{k}{n}\Big)^{k-n}\left[T_n(1-k/n)v_n\right](k)$$
$$= \sum_{k\in K_n} |\alpha_k|(n-k)!e^{n-k}\Big(1-\frac{k}{n}\Big)^{k-n} e^{k-n}\frac{(1-k/n)^{n-k}}{(n-k)!} = \sum_{k\in K_n} |\alpha_k|$$

and by Lemma 9, using $(k-l)/n \ge d_n/n \ge 2\sqrt{\log(n)/n}$ for $k, l \in K_n$ with $k \ne l$,

$$\sum_{k\in K_n} (n-k)!e^{n-k}\Big(1-\frac{k}{n}\Big)^{k-n}\Big[\sum_{l\in K_n,\ l\ne k} |\alpha_l| T_n(1-l/n)v_n\Big](k)$$
$$= \sum_{k,l\in K_n,\ k\ne l} (n-k)!e^{n-k}\Big(1-\frac{k}{n}\Big)^{k-n}|\alpha_l| e^{l-n}\frac{(1-l/n)^{n-k}}{(n-k)!} \le \frac{\log(n)}{n}\sum_{l\in K_n} |\alpha_l|.$$

That is, $\varphi(y) \ge (1-\log(n)/n)\sum_{k\in K_n} |\alpha_k|$, as stated. Thus for n sufficiently large, on Y_n the norm $\|.\|_n$ is approximately an l^1 norm.

3.2. The Spectral Mapping Theorem. Using the l^1-like subspace isolated in the previous section we show that our construction allows a drastic failure of any spectral mapping theorem. Set

$$S = \{t \in \mathbb{R}^+ \setminus \{0\} \colon nt(\sqrt{\gamma_n}\lfloor \log(n)/\sqrt[3]{\gamma_n}\rfloor)^{-1} \in \mathbb{N} \text{ for infinitely many } n\}.$$

We show that for every $t \in S$,

(24)

$$\{\lambda \in \mathbb{C} \colon |\lambda| = 1\} \subset \sigma(\mathcal{T}(t)) \text{ and } \{\lambda \in \mathbb{C} \colon |\lambda| = r\} \cap \sigma(\mathcal{T}(t)) \neq \emptyset \quad (0 \leq r < 1).$$

Since $(\gamma_n)_{n\in\mathbb{N}}$ can easily be chosen in such a way that S is dense in $\mathbb{R}^+$, in which case (24) holds for every $t > 0$, this is in striking contrast with $\sigma(\mathcal{A}) = \left\{-k/\sqrt{\gamma_k} \colon e^{61} \leq k < \infty\right\}$ proved in Proposition 17.4.

From now on n is assumed to be so large that (21) holds. Fix $t \in S$ and let first $\lambda \in \mathbb{C}$ satisfy $|\lambda| = 1$. We show that there exists $x_n \in \mathcal{X}$ with $\|x_n\| = 1$ such that $\|\mathcal{T}(t)x_n - \lambda x_n\| \to 0$ as $n \to \infty$; this clearly implies $\lambda \in \sigma(\mathcal{T}(t))$.

Following the notation of the previous section, let $d_n \in \mathbb{N}$ satisfy

$$4\sqrt{n\log(n)} + 1 < d_n < n/(6\log(n))$$

and with $b_n = d_n\lfloor \log(n)/\sqrt[3]{\gamma_n}\rfloor$ set

$$z_n = \sum_{j=1}^{\lfloor \sqrt[3]{\gamma_n}\rfloor} \lambda^j T_n\left(1 - \frac{\lfloor (n+1)/2\rfloor + jb_n}{n}\right) v_n.$$

Since $z_n \in Y_n$ we have

$$\frac{n-\log(n)}{n+\log(n)} \sum_{j=1}^{\lfloor \sqrt[3]{\gamma_n}\rfloor} |\lambda^j| \leq \|z_n\|_n \leq \sum_{j=1}^{\lfloor \sqrt[3]{\gamma_n}\rfloor} |\lambda^j| = \lfloor \sqrt[3]{\gamma_n}\rfloor.$$

Similarly,

$$\begin{aligned} T_n\left(\frac{b_n}{n}\right) z_n = &\sum_{j=1}^{\lfloor \sqrt[3]{\gamma_n}\rfloor} \lambda^j T_n\left(1 - \frac{\lfloor (n+1)/2\rfloor + (j-1)b_n}{n}\right) v_n \\ =&\lambda z_n + \lambda T_n\left(1 - \frac{\lfloor (n+1)/2\rfloor}{n}\right) v_n \\ &- \lambda^{\lfloor \sqrt[3]{\gamma_n}\rfloor + 1} T_n\left(1 - \frac{\lfloor (n+1)/2\rfloor + \lfloor \sqrt[3]{\gamma_n}\rfloor b_n}{n}\right) v_n \end{aligned}$$

hence

$$\left\|T_n\left(\frac{b_n}{n}\right)\frac{z_n}{\|z_n\|_n} - \lambda\frac{z_n}{\|z_n\|_n}\right\|_n \leq \frac{n+\log(n)}{n-\log(n)}\frac{2}{\lfloor \sqrt[3]{\gamma_n}\rfloor}. \tag{25}$$

Choose $d_n = \lfloor nt(\sqrt{\gamma_n}\lfloor \log(n)/\sqrt[3]{\gamma_n}\rfloor)^{-1}\rfloor$ and set

$$t_n = d_n\sqrt{\gamma_n}\lfloor \log(n)/\sqrt[3]{\gamma_n}\rfloor n^{-1}.$$

Then since $\gamma_n \to \infty$ as $n \to \infty$, $4\sqrt{n\log(n)}+1 < d_n < n/(6\log(n))$ holds for n sufficiently large. Let $x_n \in \mathcal{X}$ be nonzero only on X_n and $x_n|_{X_n} = z_n/\|z_n\|_n$. By $t \in S$ we have $t_n = t$ for infinitely many $n \in \mathbb{N}$ and by (25),

$$\begin{aligned}\|\mathcal{T}(t_n)x_n - \lambda x_n\| &= \left\| T_n\left(\frac{t_n}{\sqrt{\gamma_n}}\right)x_n - \lambda_n x_n\right\|_n \\ &= \left\| T_n\left(\frac{b_n}{n}\right)\frac{z_n}{\|z_n\|_n} - \lambda\frac{z_n}{\|z_n\|_n}\right\|_n \to 0 \quad \text{as } n\to\infty,\end{aligned}$$

as stated.

Let now $0 < r < 1$. We find $\lambda, \lambda_n \in \mathbb{C}$ with $|\lambda| = |\lambda_n| = r$ and $x_n \in \mathcal{X}$ $(n \in \mathbb{N})$ such that $\lambda_n \to \lambda$ and $\|\mathcal{T}(t)x_n - \lambda_n x_n\| \to 0$ as $n \to \infty$. This again implies $\lambda \in \sigma(\mathcal{T}(t))$ since $\lambda \notin \sigma(\mathcal{T}(t))$ would imply that $(\mathcal{T}(t) - \lambda')^{-1}$ $(|\lambda' - \lambda| \le \varepsilon)$ is uniformly bounded for some $\varepsilon > 0$.

Let n be such that $e^{-nt/(2\sqrt{\gamma_n})} < r$. Consider the functions

$$f_n, g_n\colon \{\lambda \in \mathbb{C}\colon |\lambda| > e^{-nt/(2\sqrt{\gamma_n})}\} \to X_n,$$

$$f_n(\lambda) = \sum_{j=-\infty}^{\lfloor\sqrt[3]{\gamma_n}\rfloor} \lambda^{j-\lfloor\sqrt[3]{\gamma_n}\rfloor-1} T_n\left(1 - \frac{\lfloor (n+1)/2\rfloor + jb_n}{n}\right)v_n,$$

$$g_n(\lambda) = \sum_{j=-\infty}^{0} \lambda^{j-1} T_n\left(1 - \frac{\lfloor (n+1)/2\rfloor + jb_n}{n}\right)v_n.$$

The coordinates of $T_n(t)v_n$ are of the form $e^{-nt}t^k/k!$ $(0 \le k \le n)$ and we have

$$b_n = d_n\lfloor \log(n)/\sqrt[3]{\gamma_n}\rfloor = \lfloor nt(\sqrt{\gamma_n}\lfloor\log(n)/\sqrt[3]{\gamma_n}\rfloor)^{-1}\rfloor\lfloor\log(n)/\sqrt[3]{\gamma_n}\rfloor \ge \frac{nt}{2\sqrt{\gamma_n}}.$$

Thus $e^{b_n}|\lambda| > 1$ for $\lambda \in \mathbb{C}$ with $|\lambda| > e^{-nt/(2\sqrt{\gamma_n})}$ so f_n and g_n are nonconstant and holomorphic on $\{\lambda \in \mathbb{C}\colon |\lambda| > e^{-nt/(2\sqrt{\gamma_n})}\}$ and they vanish at infinity. Since

$$f_n(1) - g_n(1) = \sum_{j=1}^{\lfloor\sqrt[3]{\gamma_n}\rfloor} T_n\left(1 - \frac{\lfloor (n+1)/2\rfloor + jb_n}{n}\right)v_n \in Y_n$$

implies $\|f_n(1) - g_n(1)\|_n \ge \lfloor\sqrt[3]{\gamma_n}\rfloor(n - \log(n))/(n + \log(n))$, by the maximum principle we have a $\lambda_n \in \mathbb{C}$ with $|\lambda_n| = r$ such that either $w_n = f_n(\lambda_n)$ or

$w_n = g_n(\lambda_n)$ satisfy $\|w_n\|_n \geq \sqrt[3]{\gamma_n}/3$. If $w_n = f_n(\lambda_n)$ does so then as we have seen above,

$$T_n\left(\frac{b_n}{n}\right) w_n = \sum_{j=-\infty}^{\lfloor \sqrt[3]{\gamma_n} \rfloor} \lambda_n^{j-\lfloor \sqrt[3]{\gamma_n} \rfloor -1} T_n\left(1 - \frac{\lfloor (n+1)/2 \rfloor + (j-1)b_n}{n}\right) v_n$$
$$= \lambda_n w_n - T_n\left(1 - \frac{\lfloor (n+1)/2 \rfloor + (\lfloor \sqrt[3]{\gamma_n} \rfloor) b_n}{n}\right) v_n$$

hence

$$\left\| T_n\left(\frac{b_n}{n}\right) \frac{w_n}{\|w_n\|_n} - \lambda_n \frac{w_n}{\|w_n\|_n} \right\|_n \leq \frac{3}{\sqrt[3]{\gamma_n}}.$$

If $w_n = g_n(\lambda_n)$ a similar computation gives

$$\left\| T_n\left(\frac{b_n}{n}\right) \frac{w_n}{\|w_n\|_n} - \lambda_n \frac{w_n}{\|w_n\|_n} \right\|_n \leq \frac{3}{\sqrt[3]{\gamma_n}}. \tag{26}$$

Recall that

$$d_n = \lfloor nt(\sqrt{\gamma_n} \lfloor \log(n)/\sqrt[3]{\gamma_n} \rfloor)^{-1} \rfloor \text{ and } t_n = d_n \sqrt{\gamma_n} \lfloor \log(n)/\sqrt[3]{\gamma_n} \rfloor n^{-1}.$$

Let $x_n \in \mathcal{X}$ be nonzero only on X_n and $x_n|_{X_n} = w_n/\|w_n\|_n$. By $t \in S$ we have $t_{n_k} = t$ for a subsequence $n_k \in \mathbb{N}$ $(k \in \mathbb{N})$ and by (26),

$$\|\mathcal{T}(t_n)x_n - \lambda_n x_n\| = \left\| T_n\left(\frac{t_n}{\sqrt{\gamma_n}}\right) x_n - \lambda_n x_n \right\|_n$$
$$= \left\| T_n\left(\frac{b_n}{n}\right) \frac{w_n}{\|w_n\|_n} - \lambda_n \frac{w_n}{\|w_n\|_n} \right\|_n \to 0 \quad \text{as } n \to \infty.$$

Let λ be any accumulation point of the sequence $(\lambda_{n_k})_{k\in\mathbb{N}}$. We have $|\lambda| = r$ so this choice fulfills the requirements.

Finally since $\sigma(\mathcal{T}(t))$ is closed, $0 \in \sigma(\mathcal{T}(t))$ follows. This completes the proof.

3.3. The decay of $\|R(\mathbf{i}\mu, \mathcal{A})\|$. As we have seen in (20),

$$R(\mathbf{i}\mu, \mathcal{A}) \leq \sup_{e^{61} \leq n < \infty} \min\left\{\gamma_n^{-1/2}, C_n|\mu|^{-1}\right\}.$$

With a more careful approach we could have $\gamma_n^{-1+\varepsilon}$ instead of $\gamma_n^{-1/2}$ for any $\varepsilon > 0$, which in the best case can yield

$$R(\mathbf{i}\mu, \mathcal{A}) \leq \sup_{e^{61} \leq n < \infty} \min\left\{\log(n)^{-1+\varepsilon}, C_n|\mu|^{-1}\right\}.$$

So for $|\mu| = C_n \log(n)^{1-\varepsilon}$ we get $R(\mathbf{i}\mu, \mathcal{A}) \leq \log(n)^{-1+\varepsilon}$. According to our numerical experience C_n is growing very quickly. In particular, $\|R(\mathbf{i}\cdot, \mathcal{A})\| : \mathbb{R} \to \mathbb{R}$

is far from being in any L^p class $(1 \leq p < \infty)$. It would be of interest to find an optimal sufficient condition on the decay of $\|R(\mathbf{i}\cdot, \mathcal{A})\|$ assuring the immediate norm continuity of the semigroup. As far as we know, the best result in this direction requires both $\|R(\mathbf{i}\cdot, A)\|$ and $\|R(\mathbf{i}\cdot, A^\star)\|$ to be in the same L^p class for some $1 \leq p < \infty$. Thus the construction above gives much weaker decay. However, it is important to note that it is easy to construct immediately norm continuous multiplication semigroups for which the decay of $\|R(\mathbf{i}\cdot, A)\|$ is as small as prescribed (see e.g. [7, Chapter II.4.32 p. 120]).

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